Math 436 History of Mathematics

**Discussion Problem 1:**

Consider you have just landed on Mars. You discover there is life on the planet. The people who inhabit Mars have a language of their own that you cannot understand. You observe them going about various tasks, such as eating, drinking, collecting things, etc. After 3 full days of observation and no communication, you wonder if they even know you are watching.

1. How would you determine if they have a method of counting?
2. What types of things exist on Mars?
3. How would you determine if they have a general number sense – addition, subtraction etc.?
4. What would you look for to see a notion of geometric shape and function?
5. What is the role of writing or the lack there of?

**Introduction**

As we go through this course we will try to solve problems using only what was available at the time – algorithms, number sense, algebra, geometry, etc. One of the most important tasks in this class is to **track the growth of the fundamental concepts of number sense, geometry, and algebra**. By the end of the semester we will have added calculus to this list, but only at the end. At each stage be aware of what is still lacking in mathematical concepts and why they are needed.

**UNIT 1**

**Egypt 3100 BCE**

**Watch the video:** Mathematics of ancient Egypt

<http://www.youtube.com/watch?v=Ih1ZWE3pe9o>

**Part I. Number Sense**

**Early forms of Writing**

***Hieroglyphics*** found as inscriptions on monuments, uses a left to right system, i.e 124421,1, 10, 100, etc. , ךךךئئئטטט

***Hieratic*** isciphered system found cursively on papyrus, with symbols for 1, 2, …,9, for 10 to 90 in 10s, 100 to 900 in hundreds, etc. For example 37 is 7 plus 30 written רּΔ

Information is gleaned from the Rhind Mathematical Papyrus (Ahmes) and the Moscow Papyrus, which were interpreted between 1850 and 1650 BCE. The Rind papyrus was 18 ft x 13 inches and the Moscow one was smaller, 15 ft by 3 inches.

**Zero** was used only for the bottom level in architecture.

**Algorithms for computation**:

**Multiplication of axb**: Keep doubling starting with b until you get closest to the number a you are after without going over it, keeping track of steps as you go. For example compute 18x12:

Doubling Step: number

1. 12
2. **24**

4 48

1. 96

**16** **192**  note: 32 will be over 18.

Add corresponding b column numbers: 18 = 2+16 so we add 24+192 to get 216 as the product of 18 x 12.

**Fractions**: 1/3 is written as 3 and 2/3 as 3

Writing n + m means to add 1/n to 1/m.

**Part II Algebra**

**Equations** appeared in both the Rhind and Moscow Papyruses.

For example, solve the equation 1½ n + 4 = 10

In their terminology is 2 n + 4 = 10

They would ask what multiplies by 3/2 to make 1 and then experiment to find 2/3. They then observe that they are really interested in 6 not 10 so 3 (6) = n or n = 4.

From this method of solving equations they developed **reciprocals.**

The **method of false position** is an alternative method posed for solving equations. *It is effectively trial and error. We will see this method used frequently throughout the course.*

For example: n + 4 n = 15

Try n = 4 since 4(1/4) = 1

1. + 1 + 15? No, 5 does not = 15.

How do you get 5 to 15? Multiply by 3.

So try n = 12 or 3(4) noting that 12(1/4) = 3.

12 + 3 = 15 correct

**Part III Geometry**

They knew how to compute the area of rectangles, triangles and trapezoids, similar to today’s calculations.

Great emphasis was placed on circles and areas of circles, in ways much different than today.

***Geometric Solids***: Pyramids appeared in the Moscow Papyrus, but not in the Rhind Papyrus.

They started with a truncated pyramid and discovered that the volume was V = (h/3)(a2+ab+b2). The Moscow Papyrus also considers the surface area of a sphere a 2((8/9)(d))2 for a sphere of diameter 9/2 or S = twice the area of the circle. *Compare this to the surface area of a semi-cylinder of diameter and height = 4.5*

**Babylonian (Mesopotamia)** is a bit older than the Egyptian culture having developed in about 5000BCE. Its influence ran concurrent with the Egyptian culture. The first writing ever recorded came from a temple in Uruk having been done around 4000 BCE.

**Part I Number Sense**

The first mathematics recorded was practical, to calculate the area of a field. Three different number signs to represent lengths were used. Small circles were used to denote 10 rods, a large D represented 60 rods, and a small circle with a large D represented 600 rods. A horizontal line represented width and a vertical line represented length. They found the area by finding the average width and the average length and multiplying them together.

1 square rod was 1 sar while 1800 sar were equal to 1 bur. As you can see this is the beginning of the base 60 number system used, and it arose from a measurement application. It was also the first image of a place value system.

Numerals were developed as follows:

Y represented 1 and ﮮ represented 10. The placement of the symbols corresponded to the place value system thus 3x602 + 42x60 + 9 could be written by creating gaps between the pieces and using the above symbols. An extra gap indicated zero. They used tables to do arithmetic calculations, in particular multiplication tables. No addition tables have been found. It is believed that they used partial products as we do today.

**Part II Geometry**

Geometry work preceded algebraic work and could then be used to express algebraic equations.

They were preoccupied with circles. They took the **circumference** as the defining component of a circle, not the diameter. When describing a circle they gave an ordered pair – coefficient lists computed from the circumference i.e. 0;20 means 1/3 of the circumference for the diameter and 0;05 means 1/12 of the circumference for the area. This indicates that the Babylonian value for π is the ratio of the circumference to the diameter which is 3.

The Babylonians computed the volumes of solids in much the same way as they computed areas.

Square roots and the Pythagorean Theorem grow out of their geometry.

Starting with a side of length two of a square, a tablet was found illustrating that they think that the diagonal measures 1;25 or in today’ language 1 and 5/12, corresponding to the square root of 2. There are indications that other possibilities for the square root of 2 existed, but no one knows how they were calculated.

**Part III Algebra**

One possibility links closely with algebraic expressions, including the Pythagorean Theorem. A specific case of the Pythagorean Theorem is for a square of side a and the diagonal then corresponds to a√2.

Consider a square of area N, and the Babylonians wanted a side of size √N = √a2 + b

**a c**

a

a

c Aca ac

b

**UNIT 2 Greek Part I**

**Watch the video:** Mathematics of ancient Greece

<http://www.youtube.com/watch?v=y1lIdkoIn0Y>

Timeline for Egyptian, Babylonian, and Greek

Greek influence 600 BC - 450 AD

The Greeks were traders, not farmers. The seaport was their entry to the world. They learned all they could from the Egyptian and Babylonian cultures. Economic pursuits drove their development of mathematics.

It became important to not only find solutions to problems, but to **prove** they were indeed solutions. Rational inquiry drove their development and was central to philosophy and mathematics.

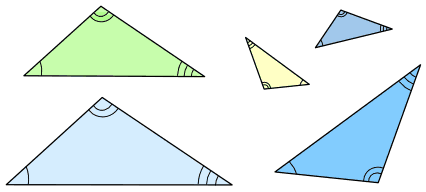
**Thales** ( a wealthy merchant) – Geometry and beginnings of trigonometry driven by problems such as:

*Determine the height of a pyramid by comparing the length of its shadow to the length of a stick of known height.*

Hieronymus (pupil of Aristotle) gave this version of Thales: *Thales noted the length of the shadow of the pyramid when it was the same length as his own shadow*.

Both renditions failed to tell how difficult it was to determine the length of the shadow. H/S = h/s; and h=s and H=S used. Only 4 days a year when H = S.

These triangles are all similar:



(Equal angles have been marked with the same number of arcs)

He also understood the notion of **congruent triangles** to measure the distance of a ship from shore using the fact that triangles are congruent if they have 2 angles and an included side equal.

Thales was first to establish (instead of intuition and experimentation) some other elementary geometric results:

1. A circle is bisected by its diameter.
2. The base angles of an isosceles triangle are equal.
3. The vertical angles formed by two intersecting lines are equal.
4. An angle inscribed in a semicircle is a right angle.

What do we mean by experiment?– vertical angles equal – draw them and cut them out and overlay them.

Thales used the proof that we use today in elementary geometry texts, using the equality of straight angles: a + c = b + c so a = b.

**Pythagoras and the Pythagoreans (his disciples):**

**Number Sense** andthe beginning of number theory**:**

The Greeks called **arithmetic** the study of abstract relationships between numbers and **logistic** the practical art of computing, which continued till the 1400s AD when the two were merged as arithmetic. Now number theory is more the early arithmetic.

Pythagoras believed that numbers (positive integers) formed the basic organizing principle for the whole universe – the basis of all physical phenomena.

He started **figurate numbers** with the split between odd and even numbers, which was depicted using pebbles, and included arithmetic of odd and even numbers He then moved on to square numbers, oblong numbers, etc.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Figurate numbers:Square Numbers | | | | | | | | |
| 1 | | | 4 | 9 | 16 | | 25 | |
| metal_marble | | | metal_marblemetal_marble metal_marblemetal_marble | metal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marble | metal_marblemetal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marble | | metal_marblemetal_marblemetal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marblemetal_marble | |
| Triangular Numbers | | | | | | | | |
| 1 | 3 | | 6 | | 10 | | 15 | |
| metal_marble | metal_marble metal_marblemetal_marble | | metal_marble metal_marblemetal_marble metal_marblemetal_marblemetal_marble | | metal_marble metal_marblemetal_marble metal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marble | | metal_marble metal_marblemetal_marble metal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marble metal_marblemetal_marblemetal_marblemetal_marblemetal_marble | |
| |  |  |  |  | | --- | --- | --- | --- | |  |  |  |  |  Oblong Numbers \*\* \*\*\* \*\*\*\* \*\*\*\*\* \*\*\*\*\*\*  \*\*\* \*\*\*\* \*\*\*\*\* \*\*\*\*\*\*  \*\*\*\* \*\*\*\*\* \*\*\*\*\*\*  \*\*\*\*\* \*\*\*\*\*\*  \*\*\*\*\*\*  2 6 12 20 30 | | | | | | | | |
|  |  | |  | |  | |  | |
|  |  | |  | |  | |  | |

A **gnomon** is the piece you add to a figurate number to go to the next larger one. Give the gnomon for triangular numbers, square numbers, and oblong numbers.

Numbers are **friendly** if each is the sum of the proper divisors of the other, i.e 220 and 284, or 1184 and 1210. Mysticism was dominant and these numbers played a role in mysticism. A **perfect number** is a number equal to the sum of its proper divisors. Name two.

**Pythagorean Triples** formulas

Constructions of Pythagorean triples:

m2 + ((m2-1)/2)2 = ((m2+1)/2)2 for odd m

or (2m)2 + (m2-1)2 = (m2+1)2 of Plato for odd or even m in 380BC

**Incommensurable numbers (*there is no common measure*) –** the beginning of **irrational numbers:**

They thought everything could be counted, including lengths. However when they considered the diagonal of a unit square they found a problem. A unit of measure could not be divided. **Aristotle,** student of **Plato,**  made the distinction between number and magnitude. Number is discrete, magnitude is continuous like lines.

*Note the use of doubling the square, and double numbers.*

They made use of what we call a proof by contradiction to indicate the existence of incommensurable numbers.

**Algebraic identities**:

For example**:** (a+b)2 = a2 + 2ab + b2

Solutions to quadratic equations as before.

**Regular Solids –** congruent faces of regular polygons

**UNIT 3 Greek II**

**Constructions**

Three famous problems:

1. (Duplication of the cube) Construct a cube whose volume is twice the original cube.

Doubling the square b2 = 2a2 a:b as b:2a

This translates to

a3:b3 as (a:b)3 and b3 = 2a3 for cubes. Find two mean proportionals such that a:b, b:c and c:2a

Hippocrates: lunes- figures bounded by the arcs of two circles.

Used to square the circle as in c.

1. (Trisection of an angle) Divide an angle into 3 equal parts
2. (Quadrature of a circle) Construct a square have an area equal to that of a given circle. Closely related to π

**Logic**

**Premises or axioms are true and well known,** also called **postulates.**

**Syllogisms** of Aristotle – if then follows

**Induction** produces new knowledge

Rules of propositions (logic)

Modus Ponens Modus Tollens Hypothetical Syllogism Alt. syllogism

p→ q p→ q p→ q pVq

p not q q→ r not p

q not p  **p → r q**

Give an example of each.

**Infinitesimals:**

Aristotle connected magnitude to our notions of infinity; begot limits and sums, but at the time no sense of convergence or infinity. Continuous line segments were ones with common endpoints.

**Zeno’s paradoxes**

Aristotle called them fallacies; it took untill the 20th century AD for Cantor, Russell and Carroll “explained” them.

*Dichotomy –* motion can’t exist

*Achilles –* Achilles can never catch the tortoise

*Arrow –* Time is made up of instants so the arrow can’t move

*Stadium – 2t = (1/2)t*

**Euclid’s *Elements* – written as a beginners text in General Mathematics**

***Elements*** were defined by the Greeks to be leading or key theorems that are of wide and general use in a subject**.** (Philosophy has its elements). They included definitions, axioms (common notions), postulates, theorems (called propositions) and proofs. There were no examples, calculations, or wit in the *Elements*.

The Greeks studied in Egypt and brought mathematics back to Greece, especially measurement, geometry, and scale models.

*Examples*:

***Definition*** *of* ***similar rectilinear figures****: are such that their angles are equal and their sides are proportional.*

***Point*** *– that which has no part.*

***Line*** *– something with breathless length*

***Angle is rectilinear*** *if it has straight sides.*

***Terminology****: to apply a parallelogram means to build one with an angle and its two sides.*

***Axiom****: Given a point P and a line l, there exists one and only one line through the point P and parallel to line l.*

***Postulate****: A circle has a center and a distance.*

***Postulate****: All right angles are equal.*

***Postulate****: If a straight line intersecting two straight lines makes the interior angles on the same side less than two right angles, then two straight lines if produced indefinitely meet on the side on which the angles are less than two right angles.*

***Axiom****: The whole is greater than a part.*

***Axiom****: If a +b = c + b then a = c.*

***Axiom****: If a = b and b = c then a = c.*

**Elements** existed prior to those of Euclid ( those of Hippocrates and Leon led to Plato’s Academy textbook), but once established they superseded all others. No book, other than the Bible, has been more widely used, edited, studied, or interpreted.

Adelard did first Latin translation from Arabic in 1120 which resides in the Vatican

Campanis translated from Greek to Latin in 1482

Billingley did first translation to English from Latin in 1570

Commandino translated from Greek to Latin in 1572 and set the stage for Simson to do the many English versions.

13 books with a total of 465 propositions.

**Book 1 and 2 are from the Pythagoreans**

**Book 1** has definitions, postulates/axioms, and 18 propositions.

1-26 are propositions relating to congruence of triangles

27-32 are propositions relating to parallels, and the sum of the angles of a triangle

33-47 are propositions relating to parallelograms, triangles, squares, and areas

48 is the converse of the Pythagorean Theorem.

**Book 2** has 14 propositions that include ones on transformations of areas and geometric algebra like the geometric proof of solutions of equations; and the generalized Pythagorean theorem, which is effectively the law of cosines:

s

rrrr r

u t

s2 = r2 + t2 + 2ru for small triangle with sides s, r, and t and obtuse angle.

**Book 3** (Sophists) has 39 propositions all on circles.

**Book 4** (Sophists) has 16 propositions all constructions with straight edge (allowed to draw a straight line of indefinite length between 2 points) and compass (allowed to draw a circle with center point and through another point). These were called and played as games, and also used as *Euclidean tools*.

***Definition:*** *To touch a circle means a line that does not intersect the circle. (tangent today)*

**Book 5** (Eudoxus) Theory of Proportions – important since true for commensurables and incommensurables.

***Definition of proportions***: Magnitudes are said to be in the same ration, the first to the second, and the third to the fourth, when if any equi-multiples what so ever be taken of the first and the third and any equi-multiples what so ever are taken of the second and the fourth, the former equi-multiples alike exceed, equal to, or less than the corresponding equi-multiples in corresponding order.

A and B are of the same kind (angles) and C and D are of the same kind (areas) so

= when mA ≥ nB as mC ≥ nD or use = or ≤.

This definition and subsequent propositions generated Dedekind’s real number system.

Book 6 applies proportions to plane geometry = similar triangles, geometric solutions to quadratic equations, propositions like the *bisector of an angle bisects the base in proportion to the other two sides.*

*.*

*a/d = b/c*

d c

a b

**Books 7-9** include 102 propositions in elementary number theory (leaves magnitudes for numbers).

It starts off with the **Euclidean algorithm** – to find the greatest common divisor of two numbers, and moving to the basic test of primality.

They include propositions on geometric progressions, the **fundamental theorem of arithmetic** that any whole number n≥ 2 is expressable as a product of primes and the proof that there are infinitely many primes.

**Book 10** is all about the irrationals, and the generation of the Pythagorean triples.

**Books 11 and 12** are on solid geometry.

**Book 13 -**  Construction of the five regular polyhedra and their representation (inscribed) in a sphere. They are the cube, tetrahedron, octahedron, dodecahedron, and the icosahedron.

[](http://upload.wikimedia.org/wikipedia/commons/c/c7/BluePlatonicDice.jpg)

**UNIT 4 Greek Part III**

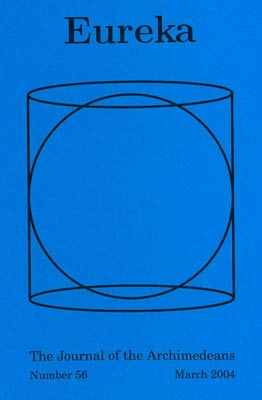
**Archimedes of Syracuse 287-212 BC (**killed at the end of the second Punic War because he would not stop what he was doing – working out a math problem.

The Fields Medal for Mathematics has his portrait on it together with the sphere inside the cylinder.

He viewed his most important work as:  
The sphere inside a cylinder has volume = 2/3 that of the cylinder and

the surface area of the sphere = 2/3 that of the cylinder.

The height of the cylinder = diameter of the sphere.



Most of Archimedes work is passed down through a Palimsest, not discovered until 1906 (a painting over the manuscript had to be removed carefully).

Geometry – circles, congruent triangles, similar triangles, spirals, etc.

Limits and the Method of Exhaustion begun with Eudoxus

Physics and Engineering – screw pump, siege machines, levers, displacement etc.

Solid Geometry

Two problems of Antiquity, one in geometry and one in physics:

***The Theorem of the Broken Chord:*** *A chord is broken into two parts AB and BC with BC greater than AB, as shown. F is on BC such that MF is perpendicular to BC and arc AM equals arc MC. Therefore F is the midpoint of the broken chord.*

M

B

A FF C

Measure off E on BC such that EC = AB In the picture E is closest to C and F closest to B.

AM = MC since they are chords that subtend equal arcs by hypothesis.

Angle A = Angle C since they are inscribed angles of the same arc.BM.

Therefore, triangle MAB = triangle MCE

And MB = ME and the triangle MBE is isosceles.

Now triangle MFB = triangle MFE with common side MF and right angles.

So BF = FE.

Combining this with AB = EC yields

BF + AB = FE + EC = FC and F is the midpoint of the broken chord.

***The Crown Problem*** – *The king commissioned a votive crown (laurel wreath) of solid gold. He worried that the contractor shorted him in gold and substituted silver for part of the gold. He wanted to know how he could catch the contractor in the act.* The story has it that Archimedes was bathing and thinking about the king’s problem and as he jumped in the bath, water jumped out (was displaced by his body). He told the king (while still in the nude) that he could submerge gold equal to what the crown should have weighed and measure the water displaced. Then measure the water displaced by the crown. If they were not equal, then the crown was not all gold. It turns out that a substitution had been made and some of the “gold” was actually silver in the crown. This was the origin of the word “Eureka”, as he exclaimed jumping nude out of the bath.

This was not Archimedes only contribution to **Physics**: He took a simple lever and put two unequal weights w1 and w2 on either end and moved the fulcrum back and forth.

a b

w1 w2

He found that the weights were inversely proportional to the distances - when the lever is balanced, a/b then w2/w1.

He also proved that 3 (3. 1408) less than C/d is less than 3 (3.1429) where C is the circumference of a circle and d is the diameter. This provides algorithms for determining the perimeter of certain regular polygons circumscribed about and inscribed in a circle, starting with regular hexagons. After getting polygons of 96 = 6 x 24 sides inside and outside, he declared he had the area of the circle.

He “counted” the grains of sand in the earth at 8x1063 by his method of exhaustion (method of discovery:

There are some, King Gelon, who think that the number of the sand is infinite in multitude; and I mean the sand not only which exists about Syracuse and the rest of Sicily, but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed this multitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the earth, including in it all the seas and the hollows of the earth filled up to a height equal to that of the highest of the mountains, would be many times further still from recognizing that any number could be expressed which exceeded the multitude of the sand so taken. But I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described, but also that of a mass equal in magnitude to the universe.

Archimedes, (c. 220 BC) letter to [Gelon II](http://www.mcs.drexel.edu/~crorres/Archimedes/Family/Gelo.html), tyrant of Syracuse.

Archimedes Calculation of π:

Measurement of a Circle

Prop: The area of a circle is equal to the area of a right triangle in which one of the legs is equal to the radius and the other to the circumference.

Proof is by exhaustion and contradiction. Inscribe polygons inside and outside of the circle and finding the areas of the polygons. Assume K is the area of the triangle and first A>K, then A<K to get a contradiction to both meaning A = K. Looks at relationship of P to K.

Prop*:* The ratio of the circumference of any circle to its diameter is less than 3 and 1/7 but greater than 3 and 10/71.

The proof ultimately determines the area of some polygons.

Among the curiosities connected to π are various mnemonics that have been devised for the purpose of remembering π to a large number of decimal places. A.C Orr published one in the Literary Digest in 1910 – replace each word by the number of letters in it:

“Now I, even I, would celebrate

In rhymes unapt, the great

Immortal Syracusan, rivaled nevermore,

Who in his wondrous lore.

Passed on before,

Left men his guidance

How to circles mensurate”.

**How many decimal places does it compute?**

About the same time another rhyme was produced:

“How I want a drink,

Alcoholic of course,

Afer the heavy lectures involving quantum mechanics”.

**How many decimal places does it compute?**

Archimedes used the Method of Exhaustion and called it his **Method of Discovery** to successively approximate the area of a circle by inscribing ever larger regular polygons, and using A1/A2 = d12/d22, to find the area of a parabolic segment by finding the sum of a geometric series, and finding the area of the Archimedean spiral.

***Quadrature of the Parabola***:

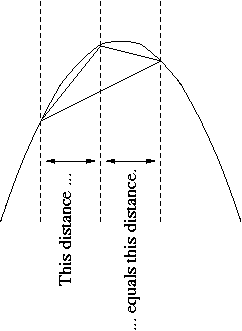
Q

R

P

R’ Q’

* **The area enclosed by a parabola and a straight line is 4/3 the area of a triangle with equal base and height.**



Create triangles PRQ and PR’Q’ from triangle PQQ’, then split each of these triangles to get a total of four triangles, etc. Let A be the area of the first triangle, then the area of the next set is ¼ that of the previous set, so we have the series:

A + ¼ A + (¼ )2A + …+ (¼)n-1 A + (¼ )n A = (4/3)A

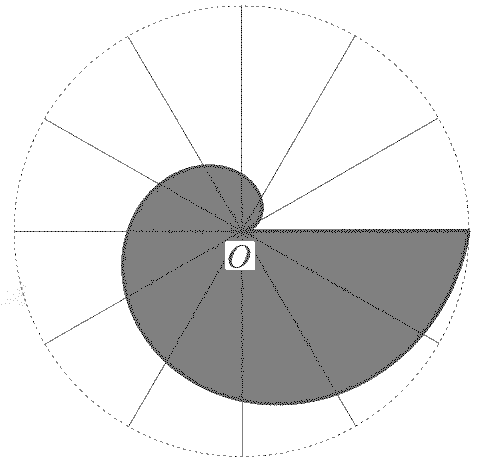
Noting that ( ¼ )n-1A = 4(¼)nA = 3( ¼ )nA + ( ¼)nA so dividing by 3 we get

1/3 ( ¼ )n-1 A = (4/3)( ¼)nA = ( ¼)nA + (1/3)( ¼)nA.

Many calculations later, we get the sum = 4/3A

**Spirals:**

**The area of the spiral is 1/3 the area of the circle (where only a revolution is considered for the spiral)**



Successive approximations by triangles yield the area of the first piece of the spiral as RA = (1/3)C, which led to the sum of the first n integral squares:

(n+1)n2 + (1+2+3+…+n) = 3(12 + 22 + 32 +…+ n2) and as a corollary:

3(12 + 22 +…+ (n-1)2) less than n3 less than 3(12 + 22 +…n2)

Cut a sphere by a plane so that the surfaces of the segments have a given ratio = a/b

B

**A A’**

B’

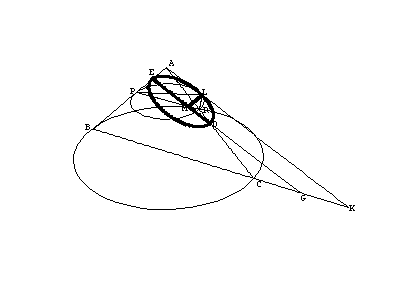
To find surfaces with areas proportional to a fixed proportion H/K

Take M so that AM/MA’ = AB2/A’B2 = circle with radius AB/circle with radius A’B= surface area of BAB’/surface area of BA’B’ = H/K

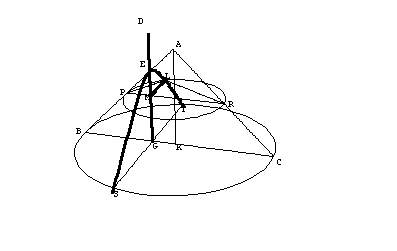
Then BAB’/BA’B’ = H/K

Definition of a parabola is the locus of points equidistant from a point called the focus and a line called the directrix.

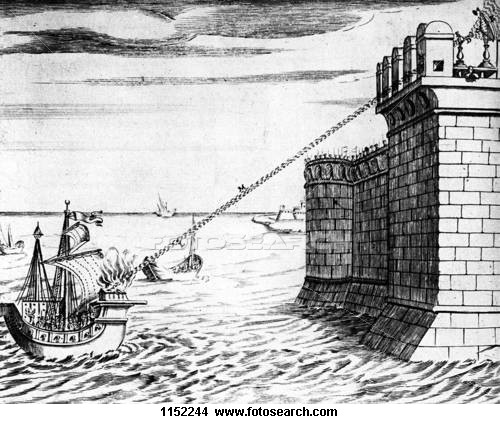
**Appolonius,** a contemporary of Archimedes, described the parabola, ellipse, and hyperbola by taking slices of the inverted cone. He described the ***symptoms*** of the curve as the relationship to the x-y plane. The following is the ellipse.

 image049.

The following is the hyperbola:

 image051.

**Focal property of the Burning Mirror** <http://history.howstuffworks.com/historical-figures/archimedes-death-ray2.htm>



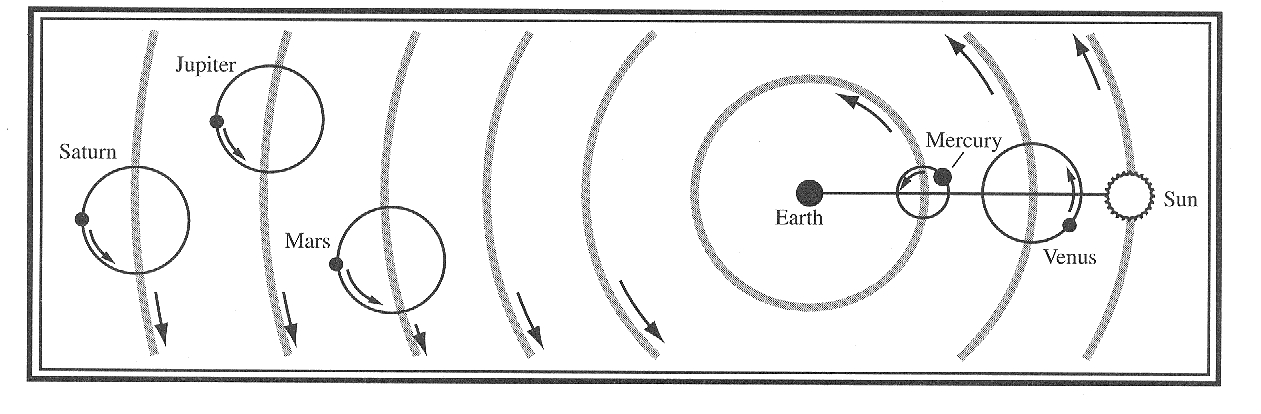
y = px2  simple equation for a parabola with axis = vertical axis and focus = (0,p) and directrix equal to the line: y = -p, where p is the distance of the vertex to the focus.

More generally, y-k = (1/4)(p)(x-h)2 with vertex (h,k) and opening up.

**Trigonometry and the Heavens**

The Greek model of the heavens consisted of spheres mixed in with star constellations. They developed properties of spheres (nothing in the Elements), such as the great circle, poles and 3 key theorems about great circles, most important of which is that any two great circles bisect each other.

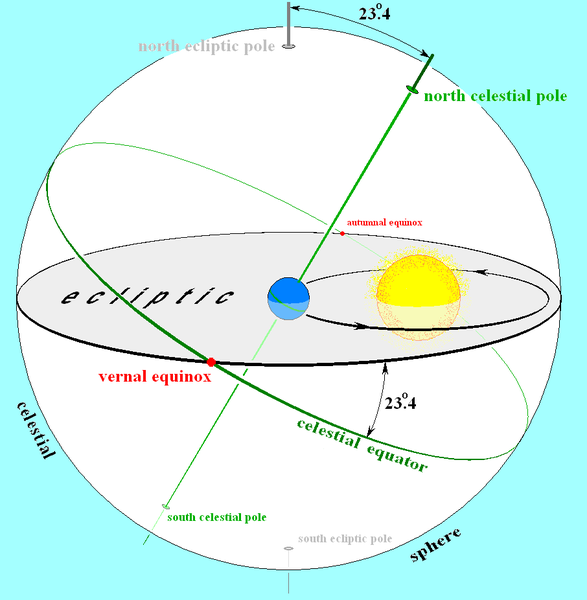
i.e. the sun’s path in a west to east movement through the stars is a great circle, passing through the 12 constellations of the zodiac.

  
Ptolemy's model for planetary motion, with deferents (big spheres) and epicycles (small spheres).

The north- south- poles diameter of the earth is the axis of rotation.

The equator and ecliptic intersect at 2 diametrically opposite points – the **vernal and autumnal equinoxes** (the sun is at those intersections). The **summer and winter solstices** are at a max distance north and south of the equator. They thought of the earth as tiny relative to the stars.

The local meridian is the great circle which passes through the north and south points of the horizon and point directly overhead, called the local zenith horizon.

[](http://upload.wikimedia.org/wikipedia/commons/b/bd/Earths_orbit_and_ecliptic.PNG)

Angle ε between the equator and the ecliptic = ½ the noon altitude of the sun at the summer and winter solstices.

Before Euclid it was thought that ε = 24 degrees, then 23 degrees 51 seconds with Ptolemy and now is 23.5 degrees.

**Ptolemy** developed a chord table 4 centuries after Archimedes, one trigonometric function, which effectively gave the plane trigonometry functions of today which come from the Islamic world in the late 800s and early 900s. He also approximated square roots to be used in his chord tables.

Chord α is the chord subtending central angle α of a circle of radius R.

Translated today, ( ½ crd α ) /R= sin (α/2) or crd α = 2Rsin(α/2)

Crd(180- α) = sq rt((2R)2-crd2 α) equivalent to sin2 + cos2 = 1.

He took the circumference of the circle as 2pr where p = 3.1418 half way between Archimedes bounds.

He was able to create crd tables in increments of 7.5 degrees and had equivalents to all of the trig identities. All numbers were written in base 60.

**Unit 5 Greek Mathematics IV**

**Greek Mathematics – or is it Mediterranean - or?**

**Syracuse Sicily King Philip of Macedonia and Alexander the Great**

**Greece**

**Athens**

**Ptolemy**

**Alexandria**

**Egypt**

**Cairo**

**Athens:** The Plato Academy 385 BC - Pythagoreans, Aristotle,

**Alexandria:** University of Alexandria: Museum and Library at Alexandria founded by Ptolemy I in 300 BC – Euclid, Archimedes, Appolonius, Pappus, and others came from Athens to Alexandria. Others were natives such as Heron and Diophantus. (many believe these two were a mixture of Greek and Egyptian heritage.) **Hypatia**, daughter of Theon, is viewed as the first woman mathematician.

**Roman Empire** in 212 BC conquered Syracuse, and by 146 BC had conquered all of Greece.

Greek mathematics, declined during this period, because the Roman imperial government did not value mathematical research and did not support it. No one wanted to learn mathematics and the *teachers* died off. Some of the Greek tradition continued in Alexandria until interest decreased there too.

**Diophantus and Greek Algebra**:

The ultimate problem solver – his major work ***Arithmetica***, is a book of 130 examples with no proofs. He introduced symbolism, powers greater than 3, used only positive rational numbers, and a general method for solving equations. He used terms like square-squares for squares multiplied by each other, such as x2y2. He was aware of rules like a minus times a minus, but did not address negative numbers. He talked about adding and subtracting likes from both sides of the equation, which also translated to multiplying and dividing. This method of using examples was called the ***Method of Problematic Analysis***.

**Sample problems:**

1. To divide a number a into two parts with a given difference b.

For a less than b, solve the equation 2x + b = a. *He used 100 and a difference of 40 to begin.*

a = 100 and b = 40: 2x + 40 = 100, 2x = 60, x = 30 and 70 for the division.

1. To divide a given number into two numbers such that given fractions of each number when added together produce a given number. *He took his fractions to be unit fractions*.

In today’s notation we are given a, b, r, s and we want to find u, v such that (u/r)+ (v/s) = b and u + v = a. He noted that (1/s)a less than b less than (1/r)a is necessary.

His example, a = 100, b = 30, r = 3, and s = 5. So we have u + v = 100 and (1/3)u + (1/5)v = 30.

He let the second part be v = 5x, so (1/5)v = x and the first part is 3(30 – x) = u.

So 5x + 3(30-x) = 100 or 5x + 90 - 3x = 100 and 2x = 10, x = 5. All of this was done without introducing symbols for anything but x.

1. To divide a given square number into two squares. (An indeterminate problem). (note he did not use squares but the words instead).

Use 16 and divide it into 2 squares. You will have x2 and 16-x2  both of which have to be squares. He chose a square of side 2x-4 or a square of 4x2 -16x+16 and set it equal to 16-x2, to get 5x2=16x. or 5x = 16, or x = 16/5, so one number is 256/25 and the other 144/25.

Translated, you get c2 = a2 + b2 , one example of the Pythagorean theorem, the circle, etc.

1. To add the same number to two given numbers so as to make each of them a square.

He took the two numbers as 2 and 3 and the required number as x. So he needed x+2 and x+3 to be squares. To solve x+2 = v2 and x+3 = u2 , an indeterminate problem. Take the difference between the two and resolve the factors: u2 – v2 = (u+v)(u-v) = 1 and noted that 4(1/4) = 1 so u+v = 4 and u-v = ¼ so 2u = 17/4 and u = 17/8 and 2v = 15/4 or v = 15/8 and x + 2 = 225/64,

Thus x = 97/64

All is not as easy as it appears however since integer values are required.

1. To find two numbers, one a square and the other a cube such that the sum of their squares is a square: (x2)2 + (y3)2 = z2. (indeterminate problem)

Showed you could take x = 2y and find 16y4 + y6 a square , to get y = 3, x = 6.

1. To divide a given square into two parts such that when we add each to the given square the remainder is a square. (Impossible)

x2 = a + b and x2 + a = c2 and x2 + b = d2

The revised format where we subtract each is possible:  
x2 = a + b and x2 – a = c2 and x2 – b = d2.

**Method of False Position**:

Recall the Egyptian version.

***Example 1***: To divide a given number into two parts such that their product is a cube minus its side.

The given number is a, so find x and y so that x + y = a, and xy = s3 – s or alternatively

y(a-y) = s3 –s, an indeterminate problem.

By example, choose a = 6 so we have y and 6-y as the two numbers or 6y – y2  = s3 – s

To make life easy he chose s = my-1 now the problem is to find the m that works. If m = 2 (the easiest) then s = 2y-1 and (2y-1)3 – 2y +1 = 8y3 -12y2 +6y -1 - 2y +1 = 8y3 – 12y2  + 4y. Note that 1 was chosen to eliminate a constant. We now have 6y – 6y2 = 8y3 -12y2 + 4y or 8y3 -6y2 -2y = 0, or 8y2 – 6y – 2 = 0 and ??

He then changed m to 3 and tried again with s = 3y – 1. We get 6y – y2 = 27y3 – 27y2 + 6y.

27y3 = 26 y2 or y = 26/27. The other part is 136/27.

For arbitrary a, we get y = (6a2 – 8)/a3

**All had the assumption that answers could be found!!**

Example: To divide unity into two parts such that if we add different numbers to each the result will be a square. He began with given the two numbers as 2 and 6 and unity = 1.

This is the only problem where he used a drawing:

2 1 6

D A G B E

Find squares close to 2 and 3

**Pappus** Proved theorems by ***Analysis*** – assume the result and get consequences until a result that is known to be true:  
p→q→r→…→a where a is known to be true, p is what you want to prove.

Now reverse to get a→…→r→q→p This only works if the converse of all r→q are true as well. Most people have ridiculed this approach.

Example: Prove: (AC + .5AB)2 = 5(.5)AB2

Given AB/AC = AC/BC

Assume truth of theorem: CD2 = 5AD2 where AD = .5AB

D A C B

CD2 = AC2 + AD2 + 2ACxAD therefore AC2 + 2ACxAD = 4AD2

But ABxAC = 2ACxAD and using the ratio gives AC2 = ABxBC.

Therefore ABxBC + AB ABxAC = 4AD2

Or AB2 = 4AD2

AB = 2AD known to be true by definition of D. Reverse the steps to get result.

**Hypatia** some consider the first woman mathematician and responsible for transcribing much of the work of Euclid and others. She is the daughter of Theon.

Unit

**Unit 6: Chinese Mathematics**

**Ancient and Medieval China**

***Number Sense in China:***

300 BCE – 100 BCE, they had two sets of symbols for , 2, …10 and alternated them to denote positioning:

through 5 then through 9

horizontal versions of the above

0 was not used as a symbol until around 100 AD

Right most column is the units and moving left for powers of 10 for a decimal place value system.

1,111 is represented by

Fractions were represented by words, for example 2/3 is written as 3 fin zhi 2, two parts from the whole of 3.

Negative numbers were denoted by colored rods, red for + and black for –

They developed the operations of +, -, x, and / for fractions.

They developed a system for computing the square root of a number

based on (x+y)2 – x2 +2xy + y2 and finding digits so that √N = 100a + 10b + c.

Use the example of √55225:

Step 1: Find the largest a such that (100a)2 ≤ N a = 2 since 40,000 ≤ 55225.

Step 2: Find the largest b such that 2(100a)(10b) + (10b)2 ≤ difference above

b such that 4000b + 100b2 ≤ 15,225. The closest b is 3.

Step 3: Now we have (100a)2 + 2(100a)(10b)+ (10b)2 = 40,000 + 12,000 + 900 = 52,900

In other words, the square root is about 230, and we only need to find the units term.

Find c so that N- 52,900 = 55225 – 52900 = 2325 ≥ 2(230). The best c you can use is c=5

Thus the square root is 235..

See picture in book which shows

(100a)2 as a center square, + borders 2( 100a)(10b) + (10b)2 and 2(c(100a+10b))

The same method applies to cube roots as well.

The Chinese solved problems and ultimately developed what we refer to as the least common denominator, when at all possible doubling to get larger denominators:

2/5 + 3/6 + 8/10 + 7/12 + 2/3: look at 5,6,10,12,and 3, 5 doubled is 10, 6 doubled is 12 and 3 doubled is 6, so 10x12 and multiply by what it takes to get 10 and or 12 respectively:

4/10 + 3/12 + 8/10 + 7/12 + 8/12 = 17/10 + 15/12 which converts to

17x12/10x12 + 15x10/10x12

As before the emphasis in geometry was on practical geometry, thus the focus was on areas and volumes, and using geometry to prove the Pythagorean Theorem.

They used four different expressions for the Area of a Circle:

½ C ½ d

Cd/4

3d2/4

C2/12

Effectively, all approximated π by 3.

They proved to themselves that this estimate was not very good by approximating the area of a circle with an inscribed and a circumscribed polygon.

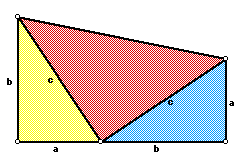
Both Liu and Zhau had geometric proofs of the Pythagorean Theorem using the replacement method. *Do many of these proofs in class.*

a

Zhao b

Liu argues the c2 = a2 + b2 = (a-b)2 + 2ab

Garfield’s proof of the Pythagorean Theorem use the area of a trapezoid:



They were the first to use matrix notation, though did not call it that, and to describe Gaussian Elimination, though they did it vertically rather than horizontally. Similarly they used synthetic division without calling it that.

They are most known for the Chinese Remainder Theorem to solve problems that could be represented by congruences.

**Example 1 of the algorithm:**

Find N the number of things such that if we count them by 3s there is a remainder of 2, if we count them by 5s the remainder is 3 and if we count them by sevens we have a remainder of 2.

N= 3x + 2 N= 5x + 3 N = 7x + 2

N = 2 mod3 N= 3mod 5 N= 2mod 7 N = rimodmi

M= 3x5x7 = 105

M1 = 105/3 = 35 M2 = 105/5 = 21 M3 = 105/7 = 15

Write each in the corresponding modulus

35 = 2mod3 21 = 1mod 5 15 – 1mod7

2x1 = 1mod3 x2 = 1mod 5 x3 = 1mod7

x1 = 5 x2 = 1 x3 = 1

N = ∑riMixi mod M = 2(35)5 + 3(21)(6) + 2(15)8 = 443mod105 = 23

***Try this next example before looking at the solution.***

***Example 2 of Algorithm****: Find N, the number of things such that if we count by 3’s the remainder is 2, by fives, the remainder is 2 and by twos the remainder is 1.*

N= 3x+2 N= 5x+ 2 N= 2x+1

N= 2mod 3 N= 2mod5 N= 1mod2 N = ri modmi

M = 3x5x2 = 30

M1 = 30/3=10 M2 = 30/5 = 6 ` M3 = 30/2 = 15

Write Mi = a(mi) + b (p = b)

10 = 3(3) + 1 6 = 1(5)+1 15 = 7(2) + 1

P1 = 1 `P2 = 1 P3 = 1

P1x1 = 1mod 3 P2x2 = 1 mod5 P3x3 = 1mod2

1x1 = 1mod3 1x2 = 1mod5 1x3 = 1mod2

X1 = 4 x2 = 6 x3 = 3

N =∑ riMixi modM = 2(10)(4) + 2(6)(6) + 1(15)(3) = 197 = 197mod30 = 17mod30.

**Unit 7: Indian Mathematics**

**India’s Contributions to Early Mathematics**:

***Number Sense***: A decimal place value system with symbols for the first nine numbers, 10 to 90, 100, 1000 and combining symbols to get larger numbers, except for 200 and 300 which had bars over the 100. By the year 600, they dropped all but the symbols for one through nine and added a symbol for 0. They could calculate square and cube roots as well as all of the other arithmetic operations, including signed numbers, and zero. There rules for division by zero seem bizarre today – namely if a/0 then a must have 0 as a divisor, thus 0/0 begins to define infinity

.

***Geometry:***The Pythagorean Theorem once again was king and used to develop many constructions, including some for circles.

Example given in class.

They had a general formula for the **Area of a quadrilateral** with sides a, b, c, d and s = ½ (a+b+c+d) as A = √(s-a)(s-b)(s-c)(s-d) and for a triangle with d= 0 called *Heron’s Rule*.

Do example with a = 3, b= 5 and c = 10 and then quadrilateral with d = 2.

***Algebra:***They developed the quadratic formula from finding the formula for the sum of an arithmetic progression: Sn = (n/2)(a + (a + (n-1)d). Given S can you find n leads to a solution of a specific quadratic equation.

They had their own variation of solving **linear congruences** by repeated substitutions. For example suppose N= 1096x + 808 and N = 3y + 0.

Then 1096x + 808 = 3y

1096 = 365(3) +1

3y -808 = [365(3) + 1]x

3y – 365(3)x = 808 + x

3(y-365x) = 808 + x

Let t = y – 365x, then 3t = 808 + x, or 3t –x = 808.

3 = 3(1) + 0 so stop and guess a value for t.

Now he says you can find a value of t that works, take t = 270, then 810-808 = 2 = x.

N = 1096(2) + 808 = 3000 and y = 1000.

The **Pell equation** could be solved using the solutions to these linear congruences:

Dx2 + or – 1 = y2, and more generally replacing 1 by b.

Use the example of Dx2 + 1 = y2 . Try (1,10) with + 8, then where 20 = 2(1)(10) and

(20,192) with + 64, and finally (120, 1151) with +1.

***Combinatorics****:* They developed the first explicit statement of combinations,

C(n,r) = n!/(r!(n-r)!), as well as many formulas for sums such as of integer squares or cubes.

***Trigonometry:***In the early 5th century they constructed approximations to the sine function and developed tables to be used in applications. (Recall the table of chords.) The first sine was that of 3.75 degrees, and all other increments of 3.75 degrees. Eventually by using half angle and sums of angles formulas they were able to calculate approximations in smaller increments. As such they also developed power series.

**Unit 8: Mathematics of Isalam and Medieval Mathematics**

**Mathematics of Islam 600-1100 CE, capitol in Baghdad, and included Spain**

**Watch the video:** [**http://www.youtube.com/watch?v=ss0k0\_ChHvU**](http://www.youtube.com/watch?v=ss0k0_ChHvU)

*Number Sense* – The decimal place value system spread from India as far as Syria by the mid seventh century. The work was translated into Arabic and became the foundation of the Hindu number system. The Moslems already had a number system so that two systems were used at the same time until the 13th century when the Arabic system (Hindu) became the standard. Arithmetic calculations began to be performed on paper, not just in the dirt, and they treated decimal fractions. One performs operations on numbers less than 1 as usual and worries about the decimal place only at the end. Initially they divided only by 2 and 10, later moving beyond these two numbers. Carrying in multiplication not there yet.

*Algebra:* One of the earliest Islamic textbook was an algebra text written in 820, which was eventually translated into Latin. Al Khwarizmi wanted a practical manual, not a theoretical one, one that provided ways of solving equations. For example, he indicated that

*Squares are equal to roots: ax2 = bx and squares and roots are equal to numbers: ax2 = c and bx = c, ax2+bx = c, ax2+ c = bx, and bx + c = ax2. There are no zero roots, and no negative numbers.*

Zero was not considered a solution to ax2 = bx. Geometric solutions similar to those of the Greeks, were used for quadratics, including establishing equations with no solutions (Ibn Turk).

Example: x2 + 10x = 39

x 5

5

x

x2 + 10x + 25 = 39 + 25

x2 + 10x + 25 = 64

(x + 5)2 = 82 so x+5 = 8 and x = 3

Example: x2 + 30 = 10xis impossible. : x2 -10x +30 and 30 >(1/2 )102

(x-5)2 + 30 -25 yields (x-5)2 + 5 = 0 and no way to depict -5 much less take the square root.-

5

x

Later, Thabit ibn Qurra and Abu Kamil introduced the work of Euclid in solving quadratic equations, further relating algebra and geometry.

Later yet, Al Karaji and al Samaw’al related algebra and arithmetic and began to work with proportions and reciprocals: 1:x as x:x2 as x2:x3 etc. and 1/x:x/x2 etc. With this iincluding square roots of polynomials, and irrational quantities. They used a table to express powers of 2, both + and – ones. With this he was able to introduce the laws of exponents: xn + xm = xn+m, and similarly from the table show that xn/xm = xn-m.

Example of long division:

Divide 20x2 + 30x by 6x2 + 12

x2 x 1

20 30

6 0 12

**Casting out nines**: The remainder of N/9 = the remainder of the (∑digits)/9

**Casting out 11s**: The remainder of N/11 = the remainder of !(S1 – S2)!/11 where S1 is the sum of the odd numbers and S2 is the sum of the even numbers and !z! is absolute value of z.

**Proof by induction** came to be and with it sidewise Pascal’s triangle and other combinatorial sums. Initially, however, geometric proofs, were provide for such things as the sum of integral squares or cubes.

Example: 13 + 23 + …+103 = (1 + 2 +…+10)2 by induction..

Activity: Prove C(n,k) = [n-(k-1)]/k x C(n,k-1) by induction on n.

The Islamic world was influenced by the Greeks and the use of inductive ideas and the application of geometry to solve algebra problems. Omar Khayyam (Al Khayyami), poet and mathematician, classified cubic equations and then proceeded to solve all types by his general method.

Consider the equation x3 + 4x = 24. If x2 = 2y, a parabola, intersects (x - 24/8)2 + y2 = (24/8)2, a semicircle, we get the required cubic.

Proof: If y = x2/2 the substituting into the semicircle equation yields (x – 3)2 + x4/4 = 9 or

x2 – 6x + 9 + x4/4 = 9, x4/4 + x2 = 6x or x3/4 + x = 6 or x3 + 4x = 24 as desired.

We can generalize to solving x3 + cx = d where x2 = (√c)y and (x - d/2c)2 + y2 = (d/2c)2 are the equations of the parabola and the semicircle respectively.

semicircle

parabola

He noted that depending on the cubic there are zero, one, two or even three solutions possible solutions.

The classification of cubics, and corresponding solutions, represented the major change in mathematical thinking in the 1300 years since Archimedes. They were now interested in finding general methods for solving all sorts of problems.

*Combinatorics*: As in Indian Mathematics, combinations C(n,r) and permutations, P(n,r) were used to count things. Now, relationships among the combinations and permutations were recognized and proved. Analogous to Pascal’s Triangle, was the table of PomPoms. How many pompoms using 10 different colors of silk are there? He noted with only one color there were 10 possibilities C(10,1). Now C(10,2) = C(1,1) + C(2,1) + … + C(9,1) = 1 +…+9 = 45. Generalizing: C(n,k) = C(k-1,k-1) + … + C(n-1,k-1). Thus achieving the table of pompoms.

Compute C(10,3) and then the rest of the table.

Recall **amicable numbers** are those such that each is the sum of the proper divisors of the other. Ibn Qurra’s theorem about amicable numbers – for n greater than 1, if pn = 3(2n-1) and

qn = 9(22n-1-1). And if pn-1, pn, qn are prime then a = 2npn-1pn and b = 2nqn are amicable. With this result, he got 220 and 284 as amicable numbers and 17, 296 and 18,416 as amicable pairs though normally attributed to Fermat.

*Geometry* – All geometry was practical. Constructions were designed for artisans and extended what the Greeks had done i.e construct a regular pentagon in a given square each of whose sides is 10. Having square roots helped – recall the construction to get lengths that are radicals. They dismissed the Greek distinction between magnitude and number.

Most famous, is the proof of Euclid’s Parallel Postulate – that if interior angles of two intersecting lines are less than two right angles, then the two straight lines will eventually meet.

The notion of parallel lines as equidistant, coupled with the sum of the angles of a quadrilateral were fundamental to the proof.

The method of exhaustion to compute volumes existed.

***Trigonometry***They, too, used trig tables and expanded them to **spherical trigonometry**, developing the counterpart to similar triangles:

If ABC and ADE are two spherical right triangles and a common angle A, then

sinBC:sinCE = sinDE:sinEA, as shown in the picture.

E

C D

A B

Dark Ages 450-1120; Period of Transition 750-1500; Modern Era 1450- present

n goes to infinity.

**Medieval Europe (1200-1400)**

**Watch the video <http://www.youtube.com/watch?v=CJ3_CBOC_dk>**

The Roman Empire collapsed in 476, and feudal societies were organized. Culture in general was low for close to 500years.

Education consisted of geometry, arithmetic, astronomy, and music. However, texts were just brief introductions.

The Monks copied Greek and Latin manuscripts, but they were not widely available. There was a huge debate on the date of Easter as there existed two calendars, the roman one based on the solar calendar and the Jewish one based on the lunar calendar.

Charlemagne liked problem solving and believed it should be the fundamental part of the arithmetic and geometry curriculum.

10th century saw a revival of mathematics through Gerhard d”Aurillac, who became Pope Sylvester III in 999. He studied in Spain where he learned from the Muslems:

Hindu Arabic Number System, abacus, the absence of zero.

1200 - translations began from Greek to Arabic, then to Latin. The Center of activity was in Toledo Spain, retaken by the Christians from the Muslims – but held repositories of Islamic Scientific manuscripts. Often a translation from Arabic to Spanish was done by a Spanish Jew and then by a Christian scholar to Latin. First books translated were those of Aristotle, Euclid, Ptolemy, Archimedes, Appollonius, and Al Khwarizmi.

**Geometry** – length of chords, arcs of a circle, with d= (s2/4h) + h where s is the length of the chord and h is the perpendicular distance to the chord, and the length of the arc was β = d/28 arc chord 28s/d). All geometry was practical, using similar triangles and right angle geometry. Leonardo of Pisa worked with chord tables and polygons inscribed in circles, and the Islamic trigonometry.

**Combinations:** Formulas for Combinations and Permutations, including factorials; and Induction to prove the formulas such as (1+2+…+n)2 = n3 + (1+2+…+(n-1))2

**Jordanus:** problems about dividing a number into 2 parts – first to establish a quadratic equation could have two roots, but still no notation, and only real positive roots. He completed the square.

**Algebra de Pisa**: *The Book of Calculations* : Chinese Remainder Theorem, method of false position, equations with multiple unknowns – problem solving.

Fibonacci Numbers 1,1,2,3,5,8,13 etc.

Quadratic formula to solve problems like xy = a and x+y = b and x2 + bx + c = 0.

X = b/2 +- √(b2/2 – c)

**Mathematics of Kinematics**: (Bradwardine and Heytesbury)early 14th century – velocity and acceleration are independent measurable quantities. They used proportions to look at velocities. If t1 = t2 the v1:v2 = s1:s2, same if s’s are equal.

Compounding of ratios exists when products of ratios are taken: a/b = b/c = c/d yields a/d.

Acceleration: Any motion is uniformly accelerated if in each of any equal parts of the time, it acquires an equal increment of velocity – velocity changing over time.

Mean Speed Rule: s = (1/2)at2 where vf = at. Oresme represented this geometrically.

Oresme’s work appeared later in the work of Galileo.

Sum of geometrical series with S = ½ a + ½ ( ½ b + …

France and Germany were in the midst of the 100 years War and the Black Death

**Unit 9: The Renaissance**

**Mathematics of the Renaissance 1300- early 1600s**

*Italy*: Italian merchants in the Middle Ages were venture capitalists, they traded with the East spurred on by the Crusades. It was a time of cultural rebirth (renaissance).

Shipbuilding industry bloomed – and with it the mathematics of perspective.

International Trading Companies grew up – they needed more sophisticated mathematics, including that applicable to business, such as double entry bookkeeping.

Money replaced barter! It was a time of commercial revolution as well.

Practical mathematics learning was added to theoretical university mathematical learning.

*Abacists* of the 1300’s wrote the texts for mathematics for a general audience, making it available to the middle class.

Trigonometry for sailing and navigation was further developed, along with logarithms, and applications to astronomy and physics.

**Algebra** dominated – the Abacists, professional mathematicians in Italy, wrote the books and were the teachers.

The knowledge of the Greek and Islamic cultures was translated from Greek and Arabic into Latin and not much later into Italian, French and German, and eventually into English.

They used the Hindu-Arabic number system with place value and algorithms for calculations, yet they held onto the Roman Numerals for legal writing (the first use of written numbers as in checkbooks). Pen and paper dominated.

They used symbols and expanded the known algebra and used abbreviations p bar for + and m bar for -. By the late 1400s the French and Germans had symbols for powers.

A most important contribution was to solutions of higher order equations:

1. ax4 –bx3 +cx2 = 0 written as x2(ax2 – bx + c) = 0 and other quartic equations

x = √((b/2a)2 + c/a) +b/2a

1. x3 + bx2 + cx = d by completing the cubes on both sides.

Ex. x3 + 60x2 + 1200x = 4000 so a = 1, b = 60, c= 1200 and d = 4000

Approximately: 100(1 + x/20)3 = 150 so (1 + x/20)3 = 1.5

1 + x/20 = cube root of (1.5)

x= 20(cube root of 1.5 – 1) = 20(1.145 – 1) = 20(.145) = 2.9

**Complex numbers**: Cardano introduced the notion, followed by Bombelli (1576-1572).

Cardano allowed negative numbers.

Bombelli wrote a systematic text, *Algebra*, widely read in Europe, where he wrote actual powers, including fractional powers, and powers of negative numbers as well as positive numbers, thus √-1 was born. He wrote the complex numbers: 2 + 3i as 2p di m3

2 – 3i as 2m di m3

He defined the arithmetic operations with complex numbers.

Examples:

2mdim3 x 2pdim3 = (2-3i)(2+3i) = 2x2 + 3x3x1 = 4 + 9 = 13

Bombelli worked with cube roots, powers expressed numerically.

*Methods of Analysis* was transmitted to Europe from the Greek and improved upon.

Rene Descartes in 1629

Francois Viete 1540-1603 who called it *Analytic Art* (similar to that of Pappus):

1. How to transform a problem into an equation
2. How to do symbolic manipulation to prove a theorem
3. How to transform an equation to find an unknown.

Simon Stevin – decimal fractions

*De Thiende* and l’Arithmetique

1. Arithmetic is the sience of numbers.
2. Number is that which explains the quantity of each thing.
3. Numbers meant different things, for example geometric numbers, commensurable numbers, square and cube numbers

**Mathematical Methods**:

***Perspective – Geometry***

First text written by Alberti in 1435, claiming that the first thing a painter needed to know was geometry. He showed how to represent a set of squares in the ground plane, which he called the *picture plane*. He saw the picture plane as a set of pierced rays of light from the various objects in the picture to the artist’s eye, whose position is called the *station point*.

**Francesca** also wrote how to draw both 2 and 3 dimensional perspectives.

**Durer** (artist- mathematician) of the same period wrote his own major treatise on the subject, published in 1525, and created a German vocabulary for scientific terms, including abstract mathematical concepts. He included such terms as intersections of circles and parabolas as well as various space curves. He used projections to get what he wanted – for example project on both the xy plane and the yz plane to get the desired effect. Durer’s own art reflected his work with perspective. He also went on to descriptions of polygons and conic sections.

**Map Making, Navigation and Geometry**

Navigation had been more by estimation than astronomy up to this point. Maps had been made since antiquity, but they preserved either distances or level of projections. Ptolemy used two different projections in his map of the world – a scale in two directions preserving ratios. However, his maps were not widely circulated in Europe. They used maps where shapes were not preserved and were elongated. Lines of compass bearings, called rhumbs, were not represented by straight lines. For small maps, this was ok, but for long voyages, more was needed. Pedro Nunes was the first mathematician to try and solve the problems, but his solution was to use spherical maps. Mercator in 1569 developed a new projection, called the Mercator projection, where both parallels and meridians were represented by straight lines on the map. Wright explained Mercator’s solution as follows:

Because the ratio of the length of a degree of longitude at latitude φ to one at the equator is equal to cos φ, if merideians are straight lines, the distances between them at latitude φ are stretched by a factor of sec φ. The stretching factor needs to vary for each small change in latitude. Let D(φ) denote the distance on the map between the equator and the parallel of latitude φ. dD = secφdφ.

**Astronomy and Trigonometry**

Regiomontanus (Mueller) – first pure trigonometry text , 1463, published 1533, specified triangle trigonometry, both plane and spherical using old methods, but with clear procedures and methods: Knowing some sides and some angles, how to get the others, using SAS congruences:

c h b

a d-a

He proved the law of sines: ratio of sides is as to the ratio of the sines of the angles.

He used these ratios to get more information, such as using equations to solve triangle equations:

D

A

B K H L G

AB/AG= AB/BD = AK/DH = sine(AGB)/sin(ABG) [sides and angles opposite them]

similarly with spherical triangles.

**Activity:** In a triangle ABC, angle A/angleB = 10/7 and angle B/angle C = 7/3. Find the 3 angles and the ratios of the sides.

Rheticus 1583 formally defined all of the trigonometric functions.

**Copernicus:** 1473-1543

Born in East Prussia, moved to Cracow Poland, then toitaly and finally to England. He “reformed” the work of Ptolemy. He did not publish his work, but Rheticus did.

Copernicus’ description of ***the universe*** was as follows:

* A system of nested spheres, the six planets, around the sun.

Sun – Mercury – Venus – Earth – Mars – Jupiter – Saturn

See picture in book.

* He knew of the earth’s rotation on its axis and the revolution about the sun but did not have a theory to explain it.
* He described the behavior of the planets based on that of the earth.
* Stars were in an outside concentric circle.
* There were only two orbits, one around the earth and one around all of the planets
* The size of the universe is huge
* He used the trigonometry and chords of Ptolemy, even though Regiomantus had provided newer versions.
* The radius of the big circle he took to be 10,000 instead of 60, used half chords and half arcs, and provided specific methods of trigonometry.
* He discovered the center is actually not the sun for either earth or the other planets, but off to the side - see picture.
* He experienced fierce opposition from the religious organizations, especially Protestant groups – they claimed his work was merely hypothesis.

**Brahe** – 1511-1553 He used observations to create many tables.

**Kepler** – 1571 - 1630Theory of elliptical orbits, and used his theological training to convince people.

* Famous quotes: “God is always a Geometer” and “The Universe is made up of numbers” (the Pythagorean doctrine)
* Kepler believed that the six planets were related to the five platonic solids.

Saturn – cube – Jupiter – tetrahedron – Mars – dodecahedron – Earth – icosohedron – Venus – octahedron – Mars

He used this relationship to determine relationships between the sizes of the orbits of the planets – each having its own.

Similarly this relationship “determined” the size of the planets: Di is diameter, s is distance and S is surface area.

**D1/D2 = 3√s and S1/S2 = (3√s)2**

* Music – he used the Pythagorean ratio of string lengths in constant harmonics to draw parallels in astronomy.
* Mysticism
* Three Laws:

1. The orbit of the earth is an ellipse with the earth at a focus
2. A planet sweeps out equal areas in equal times
3. If di = distance of a planet i to the sun then:

Period t1/period t2 = ((d1 + d2)/2)3/2

**Galileo and Kinematics**: 1564-1642 He continued the work begun in Medieval times and developed the theory of projectiles.

**Activities:**

1. Each duke has twice as many earls as dukes. Each earl has four times as many soldiers as dukes. (1/200) (sum of dukes and earls) = 9 times the number of dukes. How many dukes are there?
2. Solve x3 + 6x = 20 (Cardano)
3. Solve x4 + 3 = 12x (Ferrari)
4. Solve for x and y: xy = 8 and x2 + y2 = 27

**Unit 10: Analytic Geometry**

***17th Century Analytic Geometry***:

Modern notation began to be in use through the development of the theory of equations. Variables were used, and **x** became the most common variable used. Juxtaposition was used to denote multiplication. There was a belief that all geometry could be translated into algebra with the proper notation. Recall that previously, all algebra was done geometrically.

Both **Ferma**t and **Descartes** formalized analytic geometry, but from different vantage points. Fermat developed it from algebra to geometry, while Descartes went from geometry to algebra.

Algebraic solutions were extended to include positive and negative roots, zero, and complex numbers.

**Girard and The Fundamental Theorem of Algebra: 1629**

Girard recognized the value in the number line and the use of fractional exponents though not written in the modern way: (3/2)49 = 493/2 and 49(3/2 = 49x3/2.

He asserted first that there are as many roots of a polynomial equation as the highest power, counting multiplicities.

The roots of a polynomial equation and operations on them can be clustered as factions and this information used to link them to the coefficients of the polynomial equation, once the polynomial equation is written with powers alternating on either side of the equation , for example:

X4 – 3x3 + 2x2 + x -6 = 0 should be written as x4 + 2x2 – 6 = 3x3 - x

Let x1, x2, …, xn be the roots of the polynomial equation and 1, a1, a2, …, an be the coefficients of the polynomial, where the coefficient of the highest degree term is 1.

The first faction = x1 + … + xn = a1

The second faction = products of pairs summed = x1x2 + … + x1xn +x2x3 + … = a2

.. The last faction = x1x2…xn = an

Factions are what we call today the elementary symmetric functions.

Pascal’s Triangle tells one how many terms each faction contains. Example from above.

**Analytic Geometry**

**Fermat:** (equations to curves)

* Algebraic versions of Apollonius’ Theorems on Plane and Solid Loci: if when solving a geometric problem algebraically one ends up with two unknowns, the resulting solution is a locus (straight line or curve) , the points of which are determined by the motion of one endpoint of a variable line segment, the other of which moves along a fixed straight line. If the powers are less than or equal to 2 then one gets a straight line, circle, ellipse, or hyperbola.
* His notation had not quite evolved – he used vowels for unknowns and consonants for knowns and he still considered only positive solutions.
* He described 7 canonical forms (achieved by changing variables): (only first quadrant)

xy = b and b2 + x2 = ay2 both hyperbolas

b2 – x2 = y2 a circle

b2 – x2 = ay2 an ellipse

x2 + xy = ay2 a straight line

x2 – xy = ay2 a straight line

x2 =ay parabola

* Never published

**Descartes** (curves to equations)

* Geometric solutions to algebraic equations – coordinates – used lines and circles as in Euclid’s Elements ,but modernized the notation. I.e. z2 = az + b2

z = ½ a + √1/4 a2 + b2

picture

He used powers, such as aa = a2

* You only need to know if you can do it geometrically, such as the loci of 4 and 5 lines and then develop a machine to accomplish it. All geometric curves were defined by continuous motion.
* He only used the first quadrant.
* **Factor Theorem**: If you know one root of an equation is r then divide the polynomial by x-r to get a lower degree polynomial equation. He claimed it was **self-evident.**
* Published but it was hard to read.
* Algebra returned to the service of geometry.

Jan de Witt: worked from both directions and published the work of Fermat and understandable versions of Descartes. He also added the oblique axis, and made it perpendicular. He produced the standard forms of the circle, ellipse, and hyperbola and showed how all others could be transformed into standard form.

**The Beginnings of Probability**

The modern theory of probability is considered to have begun with the correspondence of Fermat and Pascal in 1654 about the questions posed by Antoine Gombaud:

1. How many tosses of two dice are necessary to have at least an even chance of getting a double six?
2. What is the equitable division of stakes in a game interrupted before its conclusion?

The two met in a salon in Paris in the 1650s, a hangout for mathematicians and others.

Note that with one throw the chance is 1/36 = 1/36, with two throws, it can be counted by adding getting it on the first throw to getting it on the second throw – getting it on both throws or

1/36 + 1/36 – 1/362 = 2/36 – 1/362 = 71/362 , still not even. The odds were thus 71:362-71

Because gambling dates back to earlier times, and they had some elementary understanding of how to compute odds, one believes that many cultures had a notion of probability. Jewish sources, including the Talmud, dating to early days of the common era, discuss addition and multiplication to compute the probability of compound events. Jewish law contains talk of expectation of an event, i.e when a marriage ends in death or divorce.

Dice were used in Europe in the Middle Ages, and elementary probabilities spelled out., especially where the faces of the die were used for divination. By the 1500s, the idea of equiprobable events was known and actual probability calculations could be made – a book on the Games of Chance was written in 1526 by Cardano. He used probabilities to understand odds and what one should pay to play. He was confused about multiplication for independent events – he tried to multiply odds, not probabilities, but discovered his own error. He generalized his result to: If there are f possible outcomes and s successes, the odds in favor of repetition in all n trials is: sn:fn-sn. For example if hitting a bulls eye has probability 2/7 and odds 2:6, then hitting it twice in succession is (2/7)(2/7) = 4/49 or 4:(49-4) or 4:45.

He argued that it would take half of 36 or 18 throws to get even odds of getting at least one double six – on average once every 36 rolls so for 18 throws it is 50-50. If we compute the probability of at least one occurrence, we have to consider a double appearing once, twice, three times, … up to 18 times, but this is the opposite of a double never appearing,

or 1 – (35/36)18 = .39775 or odds of .39775:.60225 not even odds, approximately 2:3, not even odds. It actually takes 25 throws to get even odds.

It was not until 1660 that many became interested in probability – to understand frequencies in chance process and as a method of determining reasonable degrees of belief. **Pascal** dealt with games of chance but in his decision theoretic argument for belief in God there was no concept of chance.

**Pascal** solved the second problem, the decision problem as follows:

He said there were always three cases:

1. If a person was to receive a fixed sum whether he wins or loses, he should receive it even if the game is halted.
2. If it is a zero-sum game and each has an equal chance of winning they should divide the stakes fifty-fifty if they are unable to complete play.
3. The division of stakes is determined by the number of games remaining and the number of games one must win to win the entire stake. Therefore if there is one more game to win to win the stake, that is all that counts. For example if the stake is $100 and player one needs one more to win and player two needs two more to win, then if player one won that next game he would get 100 and player 2 nothing; if player one did not win the next game then they would both need one more game, in which case he would get $50. So player one gets $50 + ½ (remaining $50) = $75. Similarly, all other cases are handled similarly.

Pascal’s triangle makes the computations of the other cases simpler – he called it the arithmetical triangle. He also proved a number of results about his triangle. For typing purposes we will use the notation C(n,k) to mean the kth term in the nth row, each starting with n and k = 0. Recall that the entry in the triangle is the sum of the two above it in the row above. Thus C(n,k) = C(n-1,k) + C(n-1,k-1).

C(n,k) = ∑C(j, k-1) for j= k-1 to n-1.

∑C(n,k) = 2n for k = 0 to n and C(n,k): C(n,k+1) as (k+1): (n-k)

Through the latter, he got C(n,k) = n!/(n-k)!k! Recall the Chinese had effectively the same thing. His proofs involved generalizing examples, and ultimately using induction.

He was then able to formalize the solution to the gambler’s problem:

Suppose the first player lacks r games of wining the set while the second player lacks s games and the game is interrupted at this point, then the first player should get the proportion of the total as ∑(n,k) for k – 0 to s is to 2n where n= r + s – 1 (the maximum # games left).

**Unit 11: Calculus**

**CALCULUS IS BORN**

**People:** (Some believe even Archimedes influenced the development of Calculus)

**1600’s:** Fermat, Descartes, Kepler, Torricelli, Roberval, Pascal, Von Heuraet, Gregory, Barrow

**1700’s**: Leibnitz, Newton, L’Hospital, Bernouilli, Euler, Agnesi, Lagrange

It began with a question of **Kepler**’s in 1615:

***How can one determine how much wine is left in a wine barrel?***

Kepler interpreted the solution by asking a different question:

***What is the largest parallelepiped that can be inscribed in a given sphere? Answer – a cube***

To solve the problem he calculated the volumes of many parallelepipeds inscribed in a sphere of radius 10 and found that as they tend towards the maximum, the increments are imperceptible. He used the **Method of Exhaustion**!

**Fermat** in the late 1620’s turned Kepler’s work into an algorithm and relate it to Viete’s work relating the roots of a polynomial to its coefficients, to find maximum and minimums.

Ex. x1 and x2 are roots of the equation bx – x2 = c implies that bx1 – x12 = bx2 – x22.

So b1(x1 – x2) = x12 – x22 and dividing by x1 – x2 yields **b = x1 + x2 and if x1 = x2 then x = b/2**

But recall that the original equation came from trying to divide a line of length b into two parts whose product was c.

He knew from Euclid that the maximum possible c is b2/4 and the maximum came from the two roots, each = b/2.

He applied the same method to the equation: bx2 – x3 = c and concluded that x = 2b/3 provides the maximum using geometry once again.

BUT!! How can we divide by x1 – x2 if x1 = x2? The Geometry is the same though! So he wrote the roots as x and x +e instead, and equated p(x) = p(x+e) for nonzero e.

P(x) = bx – x2 = b(x+e) – (x+e)2

Simplifying we get be = 2ex + e2 or dividing by e, we get b = 2 x+ e and if we remove e we get x = b/2.

He used the x and x+ e method to compute a tangent line. For a curve y = f(x) and a tangent line at B, pick an arbitrary point A on the tangent line and drop the perpendicular AI and BC to the axis.. Then set FI/BC = EI/CE where F is the intersection of AI with the curve. If CI = e and CD = x and CE = the subtangent can be written as f(x+e)/f(x) = (t+e)/t or tf(x+e) = (t+e)f(x). So dividing by e and removing all terms with e, there is a relationship between t and x that determines the tangent line.

A

BBB

I C D E

For example if f(x) = √x then t√(x+e) = (t+e) √x, which simplifies to t2e = 2etx + e2t, dividing through by e and then removing e terms, yields t = 2x, which is Appollonius’s result for th tangent to a parabola at double the abscissa.

**Descartes**:

Descartes was very critical of and competitive with Fermat. Recall that his approach was always geometry first. He used the fact that the radius is always perpendicular to the circumference at the point of tangency.

For P to be the center of the tangent circle, we have to have (f(x)2 + v2 – 2vx + x2 – n2 = 0 with two distinct roots. When the circle is tangent then these two roots actually coincide, so the equation has a double root. This meant that there is a factor of (x – x0)2 . So setting the above equation = (x-x0)2/q(x) and equating coefficients, Descartes could solve for v in terms of x0 and once he knew v, he could find P and the slope of the normal line.

Hudde used their algorithms to show that if p(x) has a double root, then a certain f(x) has a maximum and if it has a relative maximum M, then g(x) = f(x) – M has a double root. This ultimately led to pf(x) + bxf’(x), where p, p +b, p + 2b, is an arithmetic progression, has a double root.

**Areas and Volumes**

Kepler and Galileo add up very small pieces, such as discs, lines with no width, triangles, etc. , which they call infinitesimals or indivisibles. You should consider how we teach children to compute areas by putting the shapes on a grid and using increasingly smaller grid sizes. Similarly Archimedes’ Method of Exhaustion considered plane figures as made up of lines and solids as made up of surfaces:

**Cavalieri’s Principle**: (a student of Galileo) If two plane figures have equal altitudes made by lines parallel to the bases and at equal distances from them are in a given ratio, then the plane figures are also in the same ratio.

b

a

ab = 2∫(a/b)tdt or b2 = 2∫tdt each integral from 0 to b

The lines of the rectangle divide the triangle as well and we can get

b3 = 3∫t2dt or a2b = 3∫(a2/b2)t2dt and finally

∫xkdx = [1/(k+1)]bk+1 where the integral goes from 0 to b.

**Toricelli:** He found formulas for infinitely long solids formed by rotating xy = k2 around the y-axis from a to ∞ is finite. He computed the Volume as = to the difference of volumes of two cylinders, one horizontally below and the other vertical within.

**Fermat:** He concentrated initially on finding the area under a parabola in the first quadrant. He approximated the area with rectangles too tall and then with rectangles too short and said that the actual area was between the two. He squeezed these two values together by uniformly taking decreasing the widths of rectangles. He found a power series representation of these decreasing rectangular areas:

Px0k+1/Nk+1 (1k + 2k + …+ (n-1)k) < A < Px0k+1/Nk+1 (1k + 2k + …+ (n-1)k + nk)

∑ik-1 < Nk+1/(k+1) < ∑ik so ∑ik = Nk+1/(k+1) + Nk/2 + P where P is a polynomial in N of lesser degree.

He used this and relationships of the binomial coefficients to get

A = ∫pxkdx to be pxk+1/(k+1) or from 0 to N we get pNk+1/(k+1)

Thus, he computed the area under polynomials with positive integer coefficients and the corresponding integral.

**Wallis** extended this work to include positive fractional coefficients. To do so, he had to define **x0 = 1.**

They had more problems with the negative exponents, however, such as y = x-k.

If one uses Fermat’s rule, for k = 1, one gets 1/0 still not accepted as defined or undefined.

They gave up!!

Fermat went on to computing the area under the circle in the first quadrant by looking at

y = √(1-x2) or y = (1-x)1/2 which he generalized to y = (1-xp)n , the circle for p = n = ½

He looked at p = 2 and p = 3 and deduced that the area in question was the area under

1-3x1/2+3x-x3/2 = 1/10 using the binomial coefficients applied to fractional coefficients. The ratio is 1:1/10 = 10 which is a binomial coefficient. He then constructed a table like Pascal’s triangle, called the Arithmetic triangle, written sidewise for the values, but he needed to interpolate to get what he needed:

p/n 0 1 2 3 4

0 1 1 1 1 1

1 1 2 3 4 5

2 1 3 6 10 15

3 1 4 10 20 35

4 1 5 15 35 56

Look at row 2, column n, we get the sum of the next row of Pascal’s triangle divided by 2 which is a2n = (n+1)(n+2)/2 and a2,1/2 = ( ½ +1)( ½ +2)/2 = 15/8 and a1/2.1/2 = 4π

**St. Vincent** found the area under a hyperbola, and **Mercator** defined the log (1 + x) all using series to approximate the area under the curve, and ultimately defining specific integrals.

**Gregory and Barrow** were the first to state the Fundamental Theorem of Calculus:

Gregory used arc length to get ∫ ydx from 0 to x to be c∫√(1 + (du/dx)2) from 0 to x.

Barrow developed the inverse relationship:

∫R(x)f’(x)dx = R(f(b)-f(a)) where the integral is taken from a to b.

**Sir Issac Newton**

Newton spent 1664 to 171 on his idea of Calculus, but never published his work, though it circulated in manuscript form throughout Europe. Power series became the basis for his calculus. It enabled curves to be squared and was based on much of Wallis’ work computing the area under a circle in the first quadrant. He looked for patterns in calculating ∫(1-x2)n dx from 0 to x. He had the solution from Wallis for positive integers and fractions n and extended it to negatives using C(n,k) = n!/(k!(n-k)! and applied it to (a + bx)n by calculating C(1/2,0), C(1/2,1), … to get (1+x) = 1 + ½ x – 1/8 x2 + 1/16 x3 - …

To check, he squared the power series to get 1 + x and using the areal of y = xn as xn+1/(n+1)

We also have the area under y = 1/(1+x) = log(1+x), and as such Newton found the power series for log (1+x) = x – x2/2 + x3/3 – x4/4 + … , which allowed him to compute logs of numbers.

Newton most famous contributions are to what he called a **fluxion**, which is the speed with which x increases via its generating motion --- approximated differential equations satisfied by the curve of motion.

**Newton’s Laws of Motion**:

1. Every body preserves in its state of rest or uniform straight forward motion, unless a force is applied to it.
2. A change in motion is proportional to the force across a straight line.
3. To every action there is an equal and opposite reaction.

**Gottfried William Leibnitz –** his Calculus was based on sums and differences. If A, B,C,D,E is an increasing sequence and L, M, N, P are the successive differences, then E-A = L+M+N+P.

He produced what is called the **harmonic triangle**:

½

½ ½

1/3 1/6 1/3

¼ 1/12 1/12 ¼

1/5 1/20 1/30 1/20 1/5

Each column = quotient of first column with the corresponding column of the arithmetic triangle.

With this he derived ½ = 1/3 + 1/12 + 1/30 + 1/n(n+1)(n+2)/2 or multiplying by 3 he gets

3/2 = 1 + ¼ + 1/10 + …

He applied this to geometry:

picture

∑δyi = yn – y0 and ∑yi sequence implies that [δ∑yi] = original sequence of ordinates.

∫dy = y where δy = dy and the area under the curve is d∫y = y.

He introduced the differential d and ∫ as the area under the curve, both operators and both variables.

He had the first application of a differential to a differential triangle – an infinitesimal right triangle with hypotenuse ds.

ds:dy:dx = τ:y:t as shown in the picture, with ordinate y, tangent τ, and subtangent t.

In a circle of radius r, ydr = rdx if you replace the radius by the normal line, ydx = nds or yds = ndx or ydy = vdx with dx:dy = y:v because ∫ydy represented a triangle with area ½ b2, thus the problem of solving the equation y(dy/dx) = z is an inverse problem of tangents.

For example, the area of a quarter circle of radius 1 is as follows:

y2 = 2x – x2 or y2 + x2 -2x = 0 or y2 + (x-1)2 = 1 with center at (1,0).

∫ydx = ½ (x0y0 + ∫zdx from 0 to x0 and z + y – x(dy/dx).

z = y – x((1-x)/y) = w/y = √x/(2-x) and z2 = x/(z-x) or x = 2z2/(z2 + 1)

Quadrature of a circle:

∫ydx = 1 + ∫z2/(z2 + 1) and he got z2/(z2 + 1) = z2(1 – z2+ z4-z6 +…and

∫ydx = 1 – 1/3z3 + 1/5x5 +… equivalent to π/4

At the same time he developed rules for working with differentials:

d(a) = 0

d(v+w) = dv + dw

d(uv) = udv + vdu

d(u/v) = (vdu-udv)/v2

dz/z = xdy/y + logydx

dxn = nxn-1dx for fractional powers as well.

Also dxy is not equal to dxdy and d(x2) is not equal to (dx)2

**Fundamental Theorem and Differential Equations**

∫ydx = z(b) where dz/dx = y

He used power series to solve problems and the methods of indeterminate coefficients and was able to get a power series for x = sin y. He related the use of infinitesimals to the method of exhaustion.

Both Newton and Leibnitz accused the other of plagiarism.

Newton worked with velocity and distances, and Leibnitz worked with sums and differences.

**Euler**: 1707-1783

Euler’s ideas about differential equations led him to the current understandings of the modern notion of the sine and cosine function. Newton and Leibnitz derived differential equations for sine using geometrical arguments. The exponential function was well known to Euler early on.

Euler used series as a prerequisite to calculus and introduced it in *Inroductio,* the first of his three books on Calculus. In this book he stated: those quantities that … undergo change when others change are called functions of these quantities”, the first know use of the general word function. He did not require them to

He realized that there are two types of functions (now more) = algebraic and transcendental. The latter consisted only of exponential and logarithmic functions, which he completed the development of differentials for their solutions. Indeed he showed that

Dz – 2zdv + zdv/v = dv/v could be solved by multiplying by an integrating factor e-2vv to get the solution 2vz +1 = Ce2v. He found that higher order equations could be solved by exponential functions, but by the mid 1730s he wrote that this was not sufficient. It was not until 1739 that he realized that the sine function would enable closed form solutions of such higher order equations. He solved the differential equation of motion of a sinusoidally driven harmonic oscillator and with the resulting differential equation

2ad2s + sdt2/b + adt2/g sin (t/a) = 0 where s is position, t is time, a, b constants. He also deleted the sine and got the arcsine or the arc cosine. He solved the arccosine equation for s instead of t and got s = Ccos(t/√2ab).

Euler contributed to another area, the calculus of variations, related to differential equations. The goal here is to find the curve that maximizes or minimizes a particular integral. He was then able to combine them all into a general theory about integrals of the form

I(y) = ∫F(x,y,y’)dx from a to b.

Partial differential equations were born following this general theory, first of two variables, then of k variables. Multiple integration was employed to find solutions of partial differential equations, and then to compute volumes. The wave equation represented the partial differential equation of most interest.

Volume 2 of Euler’s Calculus books began with a definition of differential calculus. Because he believed calculus had to do with ratios of increments, he started there. He talked about finite differences: Δy = y – y’, Δy’ = y’ – y” and second differences ΔΔy = Δy’-Δy etc.

He dealt with special cases of the chain rule as they arose, rather than develop a specific rule. Most of his examples used algebraic functions, but he did use some transcendental functions.

His third volume dealt with integral calculus and begins with standard results:

∫axndx =[ a/(n+1)]xn+1 + C for n not equal to -1 and ∫adx/x = a lnx + C and moves onto

Types of integrals found in today’s texts, such as solutions by substitution and partial fractions. He then considers integration using infinite series, Newton’s favorite technique. He then considers solutions to differential equations.

**Lagrange: His** major contribution was to give a precise definition of derivative by eliminating all reference to infinitesimals. He provided the now popular definition based on a difference quotient and limits but not naming them such, using power series instead. He also contributed Taylor series and its applications.